

A Dynamical Interpretation of Connes' Unimodularity Condition in Standard Model and Majorana Neutrino

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Abstract

Standard model is minimally extended using the unitary group $G' = U(3) \times SU(2) \times U(1)$ of Connes' color-flavor algebra. In place of Connes' unimodularity condition an extra Higgs is assumed to spontaneously break G' down to standard model gauge group. It is shown that the theory becomes anomaly-free only if right-handed neutrino is present in each generation. It is also shown that the extra Higgs gives rise to large Majorana mass of right-handed neutrino and the model predicts a new vectorial neutral current.

The most promising idea of explaining small neutrino mass is the sea-saw mechanism.¹⁾ The mechanism assumes the presence of right-handed neutrino with normal Dirac mass and large Majorana mass, the mass eigenstates being Majorana neutrinos. The purpose of the present note is to propose a way of introducing right-handed neutrino and the associated large energy scale in the standard model with a dynamical interpretation of Connes' unimodularity condition.²⁾ The latter serves as a mathematical restriction^{2),3)} to reduce the unitary group $G' = U(3) \times SU(2) \times U(1)$ of Connes' color-flavor algebra²⁾ to the standard model gauge group $G_{SM} = SU(3) \times SU(2) \times U(1)$. Our dynamical interpretation of the unimodularity condition assumes extra Higgs singlet, which triggers the symmetry breakdown, $G' \rightarrow G_{SM}$, and generates an extra massive gauge boson singlet. These singlets are expected not to affect the low energy phenomenology but the extra heavy gauge boson leads to a new vector-like neutral current. It is incidentally shown that right-handed neutrino is *necessary* to achieve anomaly cancellation.

Let us first summarize Connes' reformulation of the standard model in a way convenient for later purpose. The total fermion field is represented as (bimodular) 4×4 matrix-valued spinor

$$\psi = \begin{pmatrix} l_L & q_L^r & q_L^b & q_L^g \\ l_R & q_R^r & q_R^b & q_R^g \end{pmatrix}, \quad (1)$$

where $l_L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$, $l_R = \begin{pmatrix} \nu_{eR} \\ e_R \end{pmatrix}$, $q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$, $q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}$ and r, b, g are color indices. For simplicity we consider only one generation. The gauge group in Connes' approach is taken as the unitary group of the color-flavor algebra²⁾

$$\mathcal{A}_C = C^\infty(M_4) \otimes (\mathbf{H} \oplus \mathbf{C} \oplus M_3(\mathbf{C})), \quad (2)$$

where \mathbf{H} denotes real quaternions, \mathbf{C} complex field and $M_3(\mathbf{C})$ is the set of 3×3 complex matrices so that

$$\mathcal{U}(\mathcal{A}_C) = \text{Map}(M_4, U(3) \times SU(2) \times U(1)) = G'. \quad (3)$$

The gauge transformation is induced by the unitary restriction of the algebra representation

$$\psi \rightarrow^g \psi = g\psi G, \quad (4)$$

where

$$g = \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}, \quad g_R = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad G = \begin{pmatrix} u^* & 0 \\ 0 & V^T \end{pmatrix}. \quad (5)$$

Here $g_L \in SU(2)_L$, $u = e^{i\alpha}$, α being real, and $V \in \mathcal{U}(M_3(\mathbf{C}))$ are all local. On the other hand, the gauge transformation of Higgs doublet $h = \begin{pmatrix} \phi_0^* & \phi_+ \\ -\phi_- & \phi_0 \end{pmatrix}$ reads ^{*)}

$$h \rightarrow {}^g h = g_L h g_R^\dagger. \quad (6)$$

The spontaneous breakdown of symmetry, $G_{SM} \rightarrow SU(3) \times U(1)_{em}$, is given by $g_L \rightarrow g_R$ since $\langle h \rangle = (v/\sqrt{2})\mathbf{1}_2 \neq 0$.

Since $\det g = 1$ is obeyed, Connes' unimodularity condition²⁾ is formulated as

$$\det G = 1, \quad (7)$$

which reproduces correct hypercharge assignment of quarks (see below). Putting $V = e^{i\beta}U$ with $\det U = 1$ and β being real this implies

$$-\alpha + 3\beta = 0. \quad (8)$$

However, the unimodularity condition (7) is put by hand in this level. It is desirable to *derive* it by a dynamical mechanism. A dynamical derivation would not impose the condition (8) but rather regard $-\alpha + 3\beta$ as an *independent* gauge parameter. To this end we first write Dirac Lagrangian invariant against the gauge transformation (4):

$$\mathcal{L}_D = \text{tr } \bar{\psi} i\gamma^\mu [(\partial_\mu + \mathcal{A}_\mu)\psi + \psi \mathcal{B}_\mu]. \quad (9)$$

The gauge fields transform like

$$\begin{aligned} \mathcal{A}_\mu \rightarrow {}^g \mathcal{A}_\mu &= g \mathcal{A}_\mu g^\dagger + g \partial_\mu g^\dagger, \\ \mathcal{B}_\mu \rightarrow {}^g \mathcal{B}_\mu &= G^\dagger \mathcal{B}_\mu G - G^\dagger \partial_\mu G. \end{aligned} \quad (10)$$

Putting

$$\begin{aligned} \mathcal{A}_\mu &= \begin{pmatrix} -ig_2/2 \sum_{i=1}^3 \tau_i A_\mu^i & 0 \\ 0 & -ig_1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B_\mu \end{pmatrix}, \\ \mathcal{B}_\mu &= \begin{pmatrix} +ig_1/2 B_\mu & 0 \\ 0 & -ig_3/2 \sum_{A=0}^8 \lambda_A^T G_\mu^A \end{pmatrix}, \end{aligned} \quad (11)$$

^{*)} The hypercharge of Higgs ϕ is normalized to be +1, leading to the choice $u = e^{i\alpha}$.

where τ_i ($i = 1, 2, 3$) are Pauli matrices, $\lambda_{A=a}$ ($a = 1, 2, \dots, 8$) Gell-Mann matrices with $\lambda_0 = (g'_3/g_3)1_3$,^{*)} the gauge transformation law (10) for Abelian gauge fields is cast into the form

$$\begin{aligned} B_\mu &\rightarrow {}^g B_\mu = B_\mu + (2/g_1)\partial_\mu\alpha, \\ G_\mu^0 &\rightarrow {}^g G_\mu^0 = G_\mu^0 + (2/g'_3)\partial_\mu\beta. \end{aligned} \quad (12)$$

The unimodularity condition (7) means the vanishing of the trace:

$$\text{tr} \mathcal{B}_\mu = 0 \rightarrow g_1 B_\mu - 3g'_3 G_\mu^0 = 0. \quad (13)$$

Let us now derive (13) as the low energy effective condition. This also means that the hypercharge gauge field is a mixture of B_μ and G_μ^0 . There are three neutral gauge bosons, A_μ^3 , B_μ and G_μ^0 which can mix. Among them A_μ^3 mixes at the electro-weak scale, while B_μ and G_μ^0 are assumed to mix at much higher energy scale. Hence it is sufficient to consider the neutral coupling involving B_μ and G_μ^0 for our purpose:

$$\begin{aligned} \mathcal{L}_{NC;B,G^0} &= \bar{l}_L i\gamma^\mu (-ig_1/2) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} B_\mu l_L + \bar{l}_R i\gamma^\mu (-ig_1/2) \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} B_\mu l_R \\ &+ \bar{q}_L i\gamma^\mu (-ig'_3/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} G_\mu^0 q_L \\ &+ \bar{q}_R i\gamma^\mu \left[(-ig_1/2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B_\mu + (-ig_3/2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} G_\mu^0 \right] q_R. \end{aligned} \quad (14)$$

Writing $g'_3 G_\mu^0 = (1/3)g_1 B_\mu + (g'_3 G_\mu^0 - (1/3)g_1 B_\mu)$ we separate the neutral coupling under consideration as

$$\begin{aligned} \mathcal{L}_{NC;B,G^0} &= \mathcal{L}_{NC;B} + \mathcal{L}'_{NC;Z'}, \\ \mathcal{L}_{NC;B} &= \bar{l}_L i\gamma^\mu (-ig_1/2) Y(l_L) B_\mu l_L + \bar{l}_R i\gamma^\mu (-ig_1/2) Y(l_R) B_\mu l_R \\ &+ \bar{q}_L i\gamma^\mu (-ig_1/2) Y(q_L) B_\mu q_L + \bar{q}_R i\gamma^\mu (-ig_1/2) Y(q_R) B_\mu q_R, \\ \mathcal{L}'_{NC;Z'} &= \bar{q} i\gamma^\mu (-ig_0/2) Z'_\mu q, \end{aligned} \quad (15)$$

where we have defined

$$g_0 Z'_\mu = g'_3 G_\mu^0 - (1/3)g_1 B_\mu, \quad (16)$$

^{*)} From now on we identify $G' = SU(3) \times SU(2) \times U(1) \times U(1)$. The case $g'_3 = g_3$ recovers the original G' .

and $Y(f)$ is the hypercharge 2×2 matrix of fermions $f = l_L, l_R, q_L, q_R$. The unimodularity condition (13) being imposed is tantamount to putting $Z'_\mu = 0$ by hand. The resulting Lagrangian leads to the correct hypercharge of fermions with B_μ being the hypercharge gauge field and g_1 being the hypercharge coupling constant. This is what was achieved in Connes' reformulation of standard model.²⁾

In contrast, we demand that both Z'_μ and its orthogonal field $B'_\mu \propto (1/3)g_1 G_\mu^0 + g'_3 B_\mu$ are dynamical and the latter is to be identified with the hypercharge gauge field. We write them as

$$\begin{aligned} Z'_\mu &= \cos \delta G_\mu^0 - \sin \delta B_\mu, \\ B'_\mu &= \sin \delta G_\mu^0 + \cos \delta B_\mu, \end{aligned} \quad (17)$$

where the mixing angle is given by

$$\tan \delta = \frac{g_1}{3g'_3}. \quad (18)$$

They transform like

$$\begin{aligned} Z'_\mu &\rightarrow {}^g Z'_\mu = Z'_\mu + (2/g_0)\partial_\mu(\beta - \alpha/3) = Z'_\mu - (2/g_1 \sin \delta)\partial_\mu(\alpha - \gamma), \\ B'_\mu &\rightarrow {}^g B'_\mu = B'_\mu + (2/g'_1)\partial_\mu \gamma, \quad \gamma = \alpha + 3 \sin^2 \delta(\beta - \alpha/3) \equiv \alpha + \gamma'. \end{aligned} \quad (19)$$

To make Z'_μ heavy but leave B'_μ massless we assume the presence of an extra Higgs transforming like

$$\Phi \rightarrow {}^g \Phi = e^{2i(\alpha - \gamma)} \Phi. \quad (20)$$

Nonvanishing vacuum expectation value $\langle \Phi \rangle \neq 0$ leaves $\alpha = \gamma$ unbroken. The Lagrangian of the extra Higgs singlet is given by

$$\mathcal{L}_\Phi = (D_\mu \Phi)^\dagger D^\mu \Phi - \frac{\lambda'}{4} (\Phi^\dagger \Phi - \frac{v'^2}{2})^2, \quad D_\mu \Phi = (\partial_\mu + ig_1 \sin \delta Z'_\mu) \Phi, \quad (21)$$

which gives rise to the extra gauge boson mass, $M_{Z'}^2 = g_1^2 \sin^2 \delta v'^2$, leaving B'_μ massless as desired. By inserting the inverse of (17)

$$\begin{aligned} G_\mu^0 &= \cos \delta Z'_\mu + \sin \delta B'_\mu, \\ B_\mu &= -\sin \delta Z'_\mu + \cos \delta B'_\mu, \end{aligned} \quad (22)$$

into the piece $\mathcal{L}_{NC;B}$ and defining

$$g'_1 = g_1 \cos \delta, \quad (23)$$

the piece $\mathcal{L}_{NC;B}$ contains a part, \mathcal{L}_{NC} , of the standard model Lagrangian with the hypercharge gauge field B'_μ and g'_1 being the hypercharge coupling constant. The remaining part can be combined with $\mathcal{L}_{NC;Z'}$ to yield an additional neutral coupling, $\Delta'\mathcal{L}_{NC}$, which involves only the gauge field Z'_μ . The low energy standard model structure emerges if we let Z'_μ and ν_R be heavy enough to be unobservable at low energy. We shall see below that Φ generates Majorana mass of order v' for ν_R , and, hence, we assume $v' \gg v$ so that the gauge boson Z' effectively decouples from and ν_R does not appear in the low energy spectrum. In this sense the unimodularity condition (13) is regarded as the low energy condition. Using the standard Higgs mechanism we obtain (A_μ^3 is now included)

$$\begin{aligned}\mathcal{L}_{NC} &= eA_\mu J_{em}^\mu + \frac{g_2}{\cos\theta_W} Z_\mu J_Z^\mu, \\ \Delta'\mathcal{L}_{NC} &= -\frac{g_1 \sin\delta}{2} Z'_\mu J_{Z'}^\mu - \frac{ig_0}{2} \bar{q}i\gamma^\mu Z'_\mu q,\end{aligned}\tag{24}$$

where \mathcal{L}_{NC} has the conventional form with B'_μ being the hypercharge gauge field and g'_1 the hypercharge gauge coupling constant and

$$J'_{Z'}^\mu = \bar{l}_L \gamma^\mu Y(l_L) l_L + \bar{l}_R \gamma^\mu Y(l_R) l_R + \bar{q}_L \gamma^\mu Y(q_L) q_L + \bar{q}_R \gamma^\mu Y(q_R) q_R.\tag{25}$$

Note that we have invoked the standard Higgs mechanism by employing the Lagrangian

$$\mathcal{L}_H = \text{tr} \left[(\mathcal{D}_\mu h)^\dagger \mathcal{D}^\mu h - \frac{\lambda}{4} (h^\dagger h - \frac{v^2}{2} \mathbf{1}_2)^2 \right],\tag{26}$$

where Higgs field h is assumed to transform under gauge transformation

$$h \rightarrow {}^g h = g_L h g_R^\dagger \begin{pmatrix} e^{-i\gamma'} & 0 \\ 0 & e^{i\gamma'} \end{pmatrix} \equiv g_L h g_R'^\dagger, \quad g_R' = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix},\tag{27}$$

with $\gamma' = \sin^2 \delta(3\beta - \alpha)$ and $\gamma = \alpha + \gamma'$. The covariant derivative $\mathcal{D}_\mu h$ is determined from (27) with the gauge transformation property (19). The replacement of g_R with g_R' comes from the requirement that B'_μ but not B_μ be the hypercharge gauge field and g'_1 but not g_1 be the hypercharge coupling constant. The spontaneous symmetry breakdown, $G' \rightarrow SU(3) \times U(1)_{em}$, is given by $g_L \rightarrow g_R'$ and $\gamma \rightarrow \alpha$.

The above gauge transformation of standard Higgs modifies that of fermions in order to preserve gauge invariance of Yukawa coupling. The modified gauge transformation is

$$\begin{aligned}l_L \rightarrow {}^g l_L &= e^{-i\alpha} g_L l_L, \quad \nu_R \rightarrow {}^g \nu_R = e^{-i\alpha+i\gamma} \nu_R, \quad e_R \rightarrow {}^g e_R = e^{-i\alpha-i\gamma} e_R, \\ \phi \rightarrow {}^g \phi &= e^{i\gamma} g_L \phi, \\ q_L \rightarrow {}^g q_L &= e^{i\beta} g_L U q_L, \quad u_R \rightarrow {}^g u_R = e^{i\beta+i\gamma} U u_R, \quad d_R \rightarrow {}^g d_R = e^{i\beta-i\gamma} U u_R.\end{aligned}\tag{28}$$

This is equivalent to replacing g_R in (4) and (5) with g'_R . It is apparent that triangle anomaly arising from lepton doublet running in the loop does not cancel that of quark doublet unless we perform the following *additional* gauge transformation

$$q \rightarrow^g q = e^{-i(\beta-\alpha/3)} q, \quad (29)$$

so that the quark sector in (28) is modified as

$$q_L \rightarrow^g q_L = e^{i\alpha/3} g_L U q_L, u_R \rightarrow^g u_R = e^{i\alpha/3+i\gamma} U u_R, d_R \rightarrow^g d_R = e^{i\alpha/3-i\gamma} U u_R. \quad (30)$$

The gauge transformation (29) introduces additional gauge interaction which is just negative of $\mathcal{L}'_{NC;Z'}$ in (15).

The gauge transformation of leptons given by (28) and that of quarks given by (30) finally determine the gauge interaction of fermions. By investigating non-safe triangle diagrams, $B^3, SU(3)B'^2, SU(3)SU(2)B', SU(2)B'^2, [SU(3)]^2 B', [SU(3)]^2 B$, it can be shown that the theory is anomaly-free. In particular, ν_R participates in anomaly cancellation in a non-trivial way.

The neutral coupling involving B and B' is now given by

$$\begin{aligned} \mathcal{L}_{NC;B,G^0} + \Delta\mathcal{L}_{NC;Z'} &= \mathcal{L}_{NC;B'} + \mathcal{L}_{NC;Z'}, \\ \mathcal{L}_{NC;Z'} &= (g_1 \sin \delta/2) Z'_\mu J_{Z'}^\mu, \quad J_{Z'}^\mu = [\bar{l}\gamma^\mu l - (1/3)\bar{q}i\gamma^\mu q], \end{aligned} \quad (31)$$

where $\mathcal{L}_{NC;B'}$ is the same as $\mathcal{L}_{NC;B}$ given by (15) with $B \rightarrow B'$ and $g_1 \rightarrow g'_1$ and the extra coupling $\Delta\mathcal{L}_{NC;Z'}$, which comes from non-vanishing $\gamma' = \gamma - \alpha = 3 \sin^2 \delta (\beta - \alpha/3)$ associated with the gauge field Z'_μ , see (19), is given by

$$\Delta\mathcal{L}_{NC;Z'} = (ig_1 \sin \delta/2) \bar{l}_R i\gamma^\mu Z'_\mu \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} l_R + (l_R \rightarrow q_R). \quad (32)$$

Comparing (20) and ν_R transformation property of (28) we also have the invariant coupling

$$\nu_R^T C \Phi \nu_R + \bar{\nu}_R^T C \Phi^\dagger \bar{\nu}_R, \quad (33)$$

which generates Majorana mass of order v' for ν_R . Such Yukawa couplings are absent for other fermions. Since neutrino possesses Yukawa coupling generating normal Dirac mass, the sea-saw mechanism works in this model without invoking GUT.

Unfortunately, the new energy scale v' is quite arbitrary except that it must be very large compared with v so as not to conflict with the present experiment. Detailed phenomenological calculations will be reported elsewhere. Moreover, one may argue that standard Higgs couples to the extra

Higgs singlet through $(\phi^\dagger \phi)(\Phi^\dagger \Phi)$. It turns out, however, that the renormalizability is not spoiled by assuming the absence of ϕ - Φ coupling since fermions do not couple to the additional Higgs which do not couple to the usual gauge bosons. Hence, one can assume that there exists no ϕ - Φ coupling. *) Only neutral current interaction due to Z' exchange contributes at yet unknown very high energy. The effective neutral coupling due to Z' exchange is given by

$$\mathcal{L}_{NC}^{eff} = -\frac{G'}{\sqrt{2}} J_{Z'\mu} J_{Z'}^\mu, \quad G' = \frac{g_1'^2 \sin^2 \delta}{2M_{Z'}^2} \ll G = \text{Fermi constant.} \quad (34)$$

We have presented a minimum extension of anomaly-free standard model with the gauge group $G' = SU(3) \times SU(2) \times U(1) \times U(1)$.⁴⁾ This is made possible if ν_R is assumed to be non-singlet under G' . Consequently, ν_R makes an important contribution to anomaly cancellation. A new energy scale is also introduced, which provides mass of new extra gauge boson and Majorana mass of ν_R .

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- 3) E. Alvarez, J. M. G.-Bondia and C. P. Martin, Phys. Letters **B364** (1995), 33.
- 4) The gauge group G' was previously considered as minimal extension of Weinberg-Salam model in various contexts by many authors. See, for instance, A. Davidson, Phys. Rev. **D20** (1979), 776. Our model was motivated, however, by interpreting Connes' unimodularity condition dynamically and we believe it to be new at least in that it originates from Connes' reformulation.

*) Nonetheless, there is no fundamental reason to prohibit ϕ - Φ coupling from the model.